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DETERMINATION OF ALL TRIPLY ORTHOGONAL SYSTEMS CONTAINING A FAMILY OF MINIMAL SURFACES.*

By T. H. Gronwall.

1. Introduction.

Two systems with the property indicated in the title are known:

First, the general triply orthogonal system containing a family of planes, and generated by tracing any orthogonal system of curves in a plane and letting this plane roll on a developable surface.†

Second, the system containing a family of minimal surfaces of revolution, that is, catenoids, the axis of revolution being the same for each individual surface. This system appears as the last case in the present investigation (eq. 58 and 59). It may be observed that this case falls under the first, since every triply orthogonal system containing a family of surfaces of revolution also contains a family of planes passing through the common axis of revolution. To these may be added at once the systems containing an arbitrary family of imaginary minimal cylinders represented by the equations in art. 189 of Darboux's Théorie des surfaces, vol. I, 2d ed., p. 341. This case, and that of the planes, exhaust the cases where the minimal surfaces cannot be represented by formulas (1) in § 2.

In the present paper, it is shown that besides the cases indicated there exists only one family of minimal surfaces belonging to a triply orthogonal system, namely the imaginary quartic surfaces

$$\frac{1}{432k(\rho)^2}(x+iy)^4+(x+iy)(x-iy+l(\rho))+z^2=0,$$

where $k(\rho)$ and $l(\rho)$ are arbitrary functions of the parameter ρ. In § 2, the differential equation of the problem—eq. (6)—is derived, and since the direct integration of this equation would present great difficulties, a preliminary function theoretic examination of the singular points of the equation is given in § 3, the result permitting the restriction of the solutions, a priori, to a small number of types, falling into two classes, to each of which one of the following two paragraphs is devoted. In these paragraphs function theoretic methods are used freely and enable us to keep the necessary algebraic calculations within very moderate bounds.

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* Read, in part, before the American Mathematical Society, April 1914.

† Darboux, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes, 2d ed., Paris, 1910, chapter 2.

2. The differential equation of the problem.

Disregarding, from the outset, the case of a plane, or the imaginary cylinders mentioned in the introduction, any other minimal surface may be represented by the Weierstrass formulas*

$$\begin{aligned} x + iy &= -u^2f'' + 2uf' - 2f + g'', \\ x - iy &= f'' - v^2g'' + 2vg' - 2g, \\ z &= uf'' - f' + vg'' - g', \end{aligned} \tag{1}$$

where f and g are analytic functions of u and v respectively. For a *real* minimal surface, u and v are conjugate complex variables, and f''' and g''' are conjugate functions. From (1) we obtain

$$\begin{aligned} &\frac{\partial}{\partial u}(x + iy) = -u^2f''', && \frac{\partial}{\partial u}(x - iy) = f''', && \frac{\partial z}{\partial u} = uf''', \\ &\frac{\partial}{\partial v}(x + iy) = g''', && \frac{\partial}{\partial v}(x - iy) = -v^2g''', && \frac{\partial z}{\partial v} = vg''', \end{aligned} \tag{2}$$

and

$$\begin{aligned} \frac{\partial}{\partial u}(x^2 + y^2 + z^2) &= \frac{\partial}{\partial u}[(x + iy)(x - iy) + z^2] \\ &= [-2f + (1 + uv)^2g'' - 2u(1 + uv)g' + 2u^2g]f''', \\ \frac{\partial}{\partial v}(x^2 + y^2 + z^2) &= [-2g + (1 + uv)^2f'' - 2v(1 + uv)f' + 2v^2f]g'''. \end{aligned} \tag{3}$$

We now form the partial differential equation defining the conjugate system formed by the lines of curvature on the surface (1), that is, an equation of the form

$$A\frac{\partial^2\theta}{\partial u^2} + B\frac{\partial^2\theta}{\partial u\partial v} + C\frac{\partial^2\theta}{\partial v^2} + D\frac{\partial\theta}{\partial u} + E\frac{\partial\theta}{\partial v} = 0$$

(where $A = A(u, v)$, etc.), admitting the particular solutions $x + iy$, $x - iy$, z and $x^2 + y^2 + z^2$. This equation is found to be

$$\frac{\partial}{\partial u}\left(\frac{1}{(1 + uv)^2f'''} \cdot \frac{\partial\theta}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{1}{(1 + uv)^2g'''} \cdot \frac{\partial\theta}{\partial v}\right), \tag{4}$$

as may be verified by (2) and (3) in the following manner:

First, $x + iy$ is a solution, since

$$\frac{\partial}{\partial u}\frac{-u^2}{(1 + uv)^2} = \frac{\partial^2}{\partial u\partial v}\frac{u}{1 + uv} = \frac{\partial}{\partial v}\frac{1}{(1 + uv)^2};$$

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* Darboux, Théorie des surfaces I, 2d ed., p. 340.

second, $x - iy$ is a solution, since

$$\frac{\partial}{\partial u}\frac{1}{(1+uv)^2} = \frac{\partial^2}{\partial u \partial v}\frac{v}{1+uv} = \frac{\partial}{\partial v}\frac{-v^2}{(1+uv)^2};$$

third, z is a solution, since

$$\frac{\partial}{\partial u}\frac{u}{(1+uv)^2} = \frac{\partial^2}{\partial u \partial v}\frac{-1}{1+uv} = \frac{\partial}{\partial v}\frac{v}{(1+uv)^2},$$

and finally, $x^2 + y^2 + z^2$ is a solution, since

$$\begin{aligned}\frac{\partial}{\partial u}\left(-\frac{2f}{(1+uv)^2} + g'' - \frac{2u}{1+uv}g' + \frac{2u^2}{(1+uv)^2}g\right) \\ = -\frac{\partial^2}{\partial u \partial v}\frac{2(vf+ug)}{1+uv}. \\ = \frac{\partial}{\partial v}\left(f'' - \frac{2v}{1+uv}f' + \frac{2v^2}{(1+uv)^2}f - \frac{2g}{(1+uv)^2}\right).\end{aligned}$$

If in (1) we let f and g depend also on a parameter ρ: $f = f(u, \rho)$ and $g = g(v, \rho)$, we obtain a family of minimal surfaces. Let $H d\rho$ denote the distance, at the point x, y, z, between the two surfaces corresponding to the parameters ρ and $\rho + d\rho$; then, when x, y and ρ are taken as independent variables, we obviously have

$$H = \frac{\frac{\partial z}{\partial \rho}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}},$$

and taking u, v and ρ as independent variables, this expression is transformed into

$$H = \frac{\frac{\partial(x, y, z)}{\partial(u, v, \rho)}}{\sqrt{\left(\frac{\partial(x, y)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, z)}{\partial(u, v)}\right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)}\right)^2}}.$$

Calculating H for the minimal surfaces (1), we readily obtain

$$(5) \quad H = \frac{\partial^2 f(u, \rho)}{\partial u \partial \rho} + \frac{\partial^2 g(v, \rho)}{\partial v \partial \rho} - \frac{2v}{1+uv}\frac{\partial f(u, \rho)}{\partial \rho} - \frac{2u}{1+uv}\frac{\partial g(v, \rho)}{\partial \rho}.$$

In order that the family (1) of minimal surfaces belong to a triply orthogonal system, it is necessary and sufficient that the expression (5) for H shall be a solution of the differential equation (4).[*]

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[*] Darboux, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes, 2d ed., p. 76.

Since (4) and (5) remain unchanged when u is replaced by v and f by g, the substitution of (5) in (4) gives an equation which, upon multiplication by $(1 + uv)^5$, takes the form

$$\{f, g, u, v\} = \{g, f, v, u\}, \tag{6}$$

where

$$\begin{aligned} \{f, g, u, v\} = {} & (1 + uv)^3 \frac{\partial}{\partial u}\left(\xi \frac{\partial^3 f}{\partial u^2 \partial \rho}\right) - 2v(1 + uv)^2 \xi \frac{\partial^3 f}{\partial u^2 \partial \rho} \\ & - 2v(1 + uv)^2 \frac{\partial}{\partial u}\left(\xi \frac{\partial^2 f}{\partial u \partial \rho}\right) + 6v^2(1 + uv)\xi \frac{\partial^2 f}{\partial u \partial \rho} \\ & + 2v^2(1 + uv) \frac{\partial}{\partial u}\left(\xi \frac{\partial f}{\partial \rho}\right) - 8v^3 \xi \frac{\partial f}{\partial \rho} \\ & - 2(1 + uv) \frac{\partial g}{\partial \rho} \frac{\partial \xi}{\partial u} + 8v \frac{\partial g}{\partial \rho} \xi, \end{aligned} \tag{7}$$

and

$$\xi = \xi(u, \rho) = \frac{1}{\dfrac{\partial^3 f(u, \rho)}{\partial u^3}}, \qquad \eta = \eta(v, \rho) = \frac{1}{\dfrac{\partial^3 g(v, \rho)}{\partial v^3}}. \tag{8}$$

Equation (6) is the differential equation of our problem.

3. Preliminary discussion of equation (6) and classification of possible solutions.

We begin by proving that, if f, g is a solution of (6), and we give ρ an arbitrary but constant value, either the pair of functions ξ and $\partial f/\partial \rho$ or the pair η and $\partial g/\partial \rho$ (or both pairs) has only isolated singular points. The method used will also give a first classification of the solutions.

To this purpose, we differentiate (6) four times in respect to u and four times in respect to v; after the necessary algebraic reductions, we obtain

$$\begin{aligned} & \frac{\partial^5 \xi}{\partial u^5} \cdot \frac{\partial^5 g}{\partial v^4 \partial \rho} + u \frac{\partial^5 \xi}{\partial u^5} \cdot \left(v \frac{\partial^5 g}{\partial v^4 \partial \rho} + 4 \frac{\partial^4 g}{\partial v^3 \partial \rho}\right) \\ & \qquad = \frac{\partial^5 f}{\partial u^4 \partial \rho} \cdot \frac{\partial^5 \eta}{\partial v^5} + \left(u \frac{\partial^5 f}{\partial u^4 \partial \rho} + 4 \frac{\partial^4 f}{\partial u^3 \partial \rho}\right) \cdot v \frac{\partial^5 \eta}{\partial v^5}. \end{aligned} \tag{9}$$

Writing

$$\begin{aligned} \varphi_1(u) &= \frac{\partial^5 \xi}{\partial u^5}, & \varphi_2(u) &= \frac{\partial^5 f}{\partial u^4 \partial \rho}, & \varphi_3(u) &= u \frac{\partial^5 f}{\partial u^4 \partial \rho} + 4 \frac{\partial^4 f}{\partial u^3 \partial \rho}, \\ \psi_1(v) &= \frac{\partial^5 \eta}{\partial v^5}, & \psi_2(v) &= \frac{\partial^5 g}{\partial v^4 \partial \rho}, & \psi_3(v) &= v \frac{\partial^5 g}{\partial v^4 \partial \rho} + 4 \frac{\partial^4 g}{\partial v^3 \partial \rho}, \end{aligned} \tag{10}$$

(9) takes the form

$$\varphi_1(u)\psi_2(v) + u\varphi_1(u)\psi_3(v) = \psi_1(v)\varphi_2(u) + v\psi_1(v)\varphi_3(u), \tag{11}$$

and here we may distinguish three cases:

Case I: $\varphi_1(u) = 0,\ \psi_1(v) \neq 0$.

Then (11) reduces to $\varphi_2(u) + v\varphi_3(u) = 0$ whence $\varphi_2(u) = \varphi_3(u) = 0$. $\varphi_1(u) = 0$ gives $\xi = P_4(u, \rho)$, where P_4 is a polynomial of the fourth degree in u; and $\varphi_2(u) = \varphi_3(u) = 0$ gives $\partial^4 f/\partial u^3 \partial\rho = 0$ or, by (8), $\partial\xi/\partial\rho = 0$, so that $P_4 = P_4(u)$ has its coefficients independent of ρ, and the same equation gives $\partial f/\partial\rho = P_2(u, \rho)$, a polynomial of the second degree in u. Therefore, finally,

$$\frac{\partial f}{\partial\rho} = A_0'(\rho) + A_1'(\rho)u + A_2'(\rho)u^2, \qquad \frac{\partial^3 f}{\partial u^3} = \frac{1}{P_4(u)}, \tag{12}$$

so that the only singularities of ξ and $\partial f/\partial\rho$ are at infinity The case $\psi_1(v) = 0,\ \varphi_1(u) \neq 0$ evidently reduces to case I by interchanging u and v, f and g.

Case II: $\varphi_1(u) = \psi_1(v) = 0$.

Equations (10) and (8) give immediately $\xi = P_4(u, \rho)$, $\eta = Q_4(v, \rho)$ or

$$\frac{\partial^3 f}{\partial u^3} = \frac{1}{P_4(u, \rho)}, \qquad \frac{\partial^3 g}{\partial v^3} = \frac{1}{Q_4(v, \rho)}, \tag{13}$$

where P_4 and Q_4 are polynomials of the fourth degree in u and v respectively; the coefficients, however, are not necessarily independent of ρ. The only singularities of $\partial f/\partial\rho$ and $\partial g/\partial\rho$ at finite distance are the zeros of P_4 and Q_4.

Case III: $\varphi_1(u) \neq 0,\ \psi_1(v) \neq 0$.

Writing (11) in the form

$$\frac{\psi_2(v)}{\psi_1(v)} + u\frac{\psi_3(v)}{\psi_1(v)} = \frac{\varphi_2(u)}{\varphi_1(u)} + v\frac{\varphi_3(u)}{\varphi_1(u)},$$

differentiating once in respect to v, and then giving v a constant value, it is seen that $\varphi_3(u)/\varphi_1(u)$ must be a linear function of u. Substituting this in our equation, we find the same to be true for $\varphi_2(u)/\varphi_1(u)$, so that

$$\begin{aligned} \varphi_2(u) &= (a + bu)\varphi_1(u), \\ \varphi_3(u &= (c + du)\varphi_1(u), \end{aligned} \tag{14}$$

where the constants a, b, c and d will of course depend on the parameter ρ. Substituting (14) in (11), we have

$$\begin{aligned} \psi_2(v) &= (a + cv)\psi_1(v), \\ \psi_3(v) &= (b + dv)\psi_1(v). \end{aligned} \tag{15}$$

By means of (10), equations (14) may be written

$$\frac{\partial^5 f}{\partial u^4 \partial \rho} = (a + bu)\frac{\partial^5 \xi}{\partial u^5},$$

$$u\frac{\partial^5 f}{\partial u^4 \partial \rho} + 4\frac{\partial^4 f}{\partial u^3 \partial \rho} = (c + du)\frac{\partial^5 \xi}{\partial u^5},$$

or

$$\frac{\partial^5 f}{\partial u^4 \partial \rho} = \frac{\partial^4}{\partial u^4}\left[(a + bu)\frac{\partial \xi}{\partial u}\right] - 4b\frac{\partial^4 \xi}{\partial u^4},$$

$$\frac{\partial^4}{\partial u^4}\left(u\frac{\partial f}{\partial \rho}\right) = \frac{\partial^4}{\partial u^4}\left[(c + du)\frac{\partial \xi}{\partial u}\right] - 4d\frac{\partial^4 \xi}{\partial u^4},$$

whence, integrating four times in respect to u,

$$(16)\qquad \begin{aligned} \frac{\partial f}{\partial \rho} &= (a + bu)\frac{\partial \xi}{\partial u} - 4b\xi + A_1u^3 + B_1u^2 + C_1u + D_1, \\ u\frac{\partial f}{\partial \rho} &= (c + du)\frac{\partial \xi}{\partial u} - 4d\xi + A_2u^3 + B_2u^2 + C_2u + D_2. \end{aligned}$$

The elimination of $\partial f/\partial \rho$ between these two equations gives

$$(17)\qquad \begin{aligned} [bu^2 + (a - d)u - c]\frac{\partial \xi}{\partial u} - 4(bu - d)\xi + A_1u^4 + (B_1 - A_2)u^3 \\ + (C_1 - B_2)u^2 + (D_1 - C_2)u - D_2 = 0. \end{aligned}$$

Similarly, we obtain from (15)

$$(18)\qquad \begin{aligned} \frac{\partial g}{\partial \rho} &= (a + cv)\frac{\partial \eta}{\partial v} - 4c\eta + A_3v^3 + B_3v^2 + C_3v + D_3, \\ v\frac{\partial g}{\partial \rho} &= (b + dv)\frac{\partial \eta}{\partial v} - 4d\eta + A_4v^3 + B_4v^2 + C_4v + D_4, \end{aligned}$$

$$(19)\qquad \begin{aligned} [cv^2 + (a - d)v - b]\frac{\partial \eta}{\partial v} - 4(cv - d)\eta + A_3v^4 + (B_3 - A_4)v^3 \\ + (C_3 - B_4)v^2 + (D_3 - C_4)v - D_4 = 0. \end{aligned}$$

From the first equation (16) it is clear that the only singularities of $\partial f/\partial \rho$ at finite distance are those of ξ, and these, according to the linear differential equation (17) defining ξ, cannot occur elsewhere than at the zeros of $bu^2 + (a - d)u - c$, or, should this expression vanish identically, at the zero of $bu - d$. The same argument applies to η and $\partial g/\partial \rho$. The argument fails, however, when also $bu - d$ vanishes identically; but then

$$a = b = c = d = 0,$$

and from (17) and (19) we obtain $A_1 = A_3 = 0$, and (16) and (18) give

$$\tag{20} \begin{aligned} \frac{\partial f}{\partial \rho} &= B_1 u^2 + C_1 u + D_1, \\ \frac{\partial g}{\partial \rho} &= B_3 v^2 + C_3 v + D_3. \end{aligned}$$

From these expressions we calculate H by (5), obtaining

$$\tag{21} H = -(C_1 + C_3) + 2 \cdot \frac{C_1 + C_3 + (B_1 - D_3)u + (B_3 - D_1)v}{1 + uv}.$$

It is seen at once that in order that either $\partial H/\partial u$ or $\partial H/\partial v$ shall vanish identically, H must also be identically zero; a case evidently to be excluded, since then (1) defines a single surface only, and not a family. Therefore substituting (21) in (4), we obtain

$$\xi = \frac{(1 + uv)^2}{\dfrac{\partial H}{\partial u}} \int \frac{\partial}{\partial v}\left(\frac{\eta}{(1 + uv)^2} \cdot \frac{\partial H}{\partial v}\right) du,$$

and giving v a constant value for which η is holomorphic and $\partial H/\partial u$ does not vanish, it is evident that this expression for ξ has only a finite number of singularities, and a similar conclusion applies to η. Our proposition is thus proved in all three cases.

We now proceed to deduce certain functional equations connecting $\partial f/\partial \rho$ and ξ with $\partial g/\partial \rho$ and η, and thereby obtain a second classification of possible solutions.

Equation (6) contains only the four functions referred to (and their derivatives), and since for a given value of ρ either $\partial f/\partial \rho$ and ξ have only isolated singular points, or else $\partial g/\partial v$ and η have the same property, it is evidently possible to find two points u_0 and v_0 for which $1 + u_0 v_0 = 0$ and such that $\partial f/\partial \rho$ and ξ are holomorphic for $|u - u_0| < 2r$, $\partial g/\partial \rho$ and η holomorphic for $|v - v_0| < 2r$, where r is sufficiently small.*

Consequently, for $|u - u_0| < r$ and $|h|$ sufficiently small, the value of v defined by

$$v = -\frac{1}{u} + h,$$

will be such that $|v - v_0| < 2r$, and we may substitute this value of v in (6) and expand both sides in powers of h. This gives

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* Since one of the two points u_0, v_0 is inside and the other outside the unit circle, this statement would be erroneous if, for instance, both f and g existed inside the unit circle only. Hence the necessity of the preceding investigation.

$$\cdots + 8h\xi(u, \rho)\frac{\partial^2 f(u, \rho)}{\partial u \partial \rho} + \frac{2h}{u}\frac{\partial f(u, \rho)}{\partial \rho}\frac{\partial \xi(u, \rho)}{\partial u} + 8\left(\frac{1}{u^3} - \frac{3h}{u^2}\right)\xi(u, \rho)\frac{\partial f(u, \rho)}{\partial \rho}$$

$$- 2uh\frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho}\frac{\partial \xi(u, \rho)}{\partial u} + 8\left(-\frac{1}{u} + h\right)\xi(u, \rho)$$

$$\times\left[\frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho} + h\frac{\partial^2 g\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)\partial \rho} + \cdots\right]$$

$$= \cdots + 8u^3 h\eta\left(-\frac{1}{u}, \rho\right)\frac{\partial^2 g\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)\partial \rho} + 2u^3 h\frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho}\frac{\partial \eta\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)}$$

$$- 8u^3\left[\eta\left(-\frac{1}{u}, \rho\right) + \frac{\partial \eta\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)}h + \cdots\right]$$

$$\times\left[\frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho} + \frac{\partial^2 g\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)\partial \rho}h + \cdots\right]$$

$$- 2uh\frac{\partial f(u, \rho)}{\partial \rho}\frac{\partial \eta\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)} + 8u\frac{\partial f(u, \rho)}{\partial \rho}$$

$$\times\left[\eta\left(-\frac{1}{u}, \rho\right) + \frac{\partial \eta\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)}h + \cdots\right],$$

where the terms of second and higher order in h are not written out.

Identifying the terms independent of h on both sides, we obtain

$$\left[\frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho} - \frac{1}{u^2}\frac{\partial f(u, \rho)}{\partial \rho}\right]\left[\eta\left(-\frac{1}{u}, \rho\right) - \frac{1}{u^4}\xi(u, \rho)\right] = 0. \tag{22}$$

Identifying the coefficients of h on both sides, and replacing differentiations in respect to $-(1/u)$ by such in respect to u, so that

$$\frac{\partial\eta\left(-\frac{1}{u},\rho\right)}{\partial\left(-\frac{1}{u}\right)}=u^2\frac{\partial\eta\left(-\frac{1}{u},\rho\right)}{\partial u},\qquad \frac{\partial^2 g\left(-\frac{1}{u},\rho\right)}{\partial\left(-\frac{1}{u}\right)\partial\rho}=u^2\frac{\partial^2 g\left(-\frac{1}{u},\rho\right)}{\partial u\partial\rho},$$

we find

$$\begin{aligned}(23)\qquad &\frac{4}{u}\xi(u,\rho)\frac{\partial^2 f(u,\rho)}{\partial u\partial\rho}+\frac{1}{u}\frac{\partial f(u,\rho)}{\partial\rho}\frac{\partial\xi(u,\rho)}{\partial u}-\frac{12}{u^2}\xi(u,\rho)\frac{\partial f(u,\rho)}{\partial\rho}\\ &\quad-u\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}\frac{\partial\xi(u,\rho)}{\partial u}-4u\xi(u,\rho)\frac{\partial^2 g\left(-\frac{1}{u},\rho\right)}{\partial u\partial\rho}\\ &\quad+4\xi(u,\rho)\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}+3u^5\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}\frac{\partial\eta\left(-\frac{1}{u},\rho\right)}{\partial u}\\ &\quad-3u^3\frac{\partial f(u,\rho)}{\partial\rho}\frac{\partial\eta\left(-\frac{1}{u},\rho\right)}{\partial u}=0.\end{aligned}$$

Supposing the first factor in (22) equal to zero, we have

$$\frac{\partial^2 g\left(-\frac{1}{u},\rho\right)}{\partial u\partial\rho}=\frac{1}{u^2}\frac{\partial^2 f(u,\rho)}{\partial u\partial\rho}-\frac{2}{u^3}\frac{\partial f(u,\rho)}{\partial\rho}$$

and (23) reduces to an identity.

On the other hand, when the second factor in (22) vanishes, but not the first, we have

$$\frac{\partial\eta\left(-\frac{1}{u},\rho\right)}{\partial u}=\frac{1}{u^4}\frac{\partial\xi(u,\rho)}{\partial u}-\frac{4}{u^5}\xi(u,\rho)$$

and (23) becomes

$$\begin{aligned}&\frac{\partial\xi(u,\rho)}{\partial u}\left[u^2\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}-\frac{\partial f(u,\rho)}{\partial\rho}\right]\\ &\quad-2\xi(u,\rho)\left[u^2\frac{\partial^2 g\left(-\frac{1}{u},\rho\right)}{\partial u\partial\rho}+2u\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}-\frac{\partial^2 f(u,\rho)}{\partial u\partial\rho}\right]=0.\end{aligned}$$

Since the first factor in (22) does not vanish, this may be written

$$\frac{1}{2}\frac{\partial\log\xi(u,\rho)}{\partial u}=\frac{\partial}{\partial u}\log\left[u^2\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}-\frac{\partial f(u,\rho)}{\partial\rho}\right],$$

whence by integration

$$(24) \qquad u^2 \frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho} - \frac{\partial f(u, \rho)}{\partial \rho} = \Omega(\rho) \cdot \sqrt{\xi(u, \rho)},$$

where $\Omega(\rho) \neq 0$.

According to (22) and (24), we may now distinguish two cases: A and B.

Case A:

$$(25) \qquad \frac{\partial f(u, \rho)}{\partial \rho} - u^2 \frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho} = 0.$$

Substituting v for $-(1/u)$, this may obviously also be written

$$(25) \qquad \frac{\partial g(v, \rho)}{\partial \rho} - v^2 \frac{\partial f\left(-\frac{1}{v}, \rho\right)}{\partial \rho} = 0.$$

From these equations an important conclusion may be drawn at once. Let us start at the point u_0 with a certain branch of the function $f(u, \rho)$ and continue it analytically along a closed path, returning to u_0 with a branch $f_1(u, \rho)$ of our function; then, since (6) subsists also for the branch f_1, and the continuation process does not affect the function g, it follows that (25) subsists also for the branch f_1, so that

$$\frac{\partial f_1(u, \rho)}{\partial \rho} = \frac{\partial f(u, \rho)}{\partial \rho},$$

and a similar argument applies to (25). *Therefore* $\partial f(u, \rho)/\partial \rho$ *and* $\partial g(v, \rho)/\partial \rho$ *are uniform functions of* u *and* v. The case A will be examined in § 4; it will be found convenient to subdivide it into three cases AI, AII and $AIII$ corresponding to the three cases of our first classification.

Case B:

$$(26) \qquad \eta\left(-\frac{1}{u}, \rho\right) - \frac{1}{u^4} \xi(u, \rho) = 0,$$

$$(24) \qquad u^2 \frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial \rho} - \frac{\partial f(u, \rho)}{\partial \rho} = \Omega(\rho) \sqrt{\xi(u, \rho)}, \quad \Omega(\rho) \neq 0.$$

The last equation determines, rather completely, the form of $\xi(u, \rho)$. In fact, differentiating it in respect to u, we have

$$2u\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}+u^2\frac{\partial^2 g\left(-\frac{1}{u},\rho\right)}{\partial\left(-\frac{1}{u}\right)\partial\rho}\cdot\frac{1}{u^2}-\frac{\partial^2 f(u,\rho)}{\partial u\partial\rho}=\Omega(\rho)\frac{\partial\sqrt{\xi}}{\partial u},$$

or replacing $\frac{\partial g\left(-\frac{1}{u},\rho\right)}{\partial\rho}$ by its value from (24),

$$\frac{\partial^2 g\left(-\frac{1}{u},\rho\right)}{\partial\left(-\frac{1}{u}\right)\partial\rho}=\frac{\partial^2 f(u,\rho)}{\partial u\partial\rho}-\frac{2}{u}\frac{\partial f(u,\rho)}{\partial\rho}+\Omega(\rho)\left[\frac{\partial\sqrt{\xi}}{\partial u}-\frac{2}{u}\sqrt{\xi}\right].$$

Differentiating again and multiplying by u^2, we find

$$\frac{\partial^3 g\left(-\frac{1}{u},\rho\right)}{\partial\left(-\frac{1}{u}\right)^2\partial\rho}=u^2\frac{\partial^3 f(u,\rho)}{\partial u^2\partial\rho}-2u\frac{\partial^2 f(u,\rho)}{\partial u\partial\rho}+2\frac{\partial f(u,\rho)}{\partial\rho}$$
$$+\Omega(\rho)\left[u^2\frac{\partial^2\sqrt{\xi}}{\partial u^2}-2u\frac{\partial\sqrt{\xi}}{\partial u}+2\sqrt{\xi}\right],$$

and repeating the same procedure

$$\frac{\partial^4 g\left(-\frac{1}{u},\rho\right)}{\partial\left(-\frac{1}{u}\right)^3\partial\rho}=u^4\frac{\partial^4 f(u,\rho)}{\partial u^3\partial\rho}+\Omega(\rho)u^4\frac{\partial^3\sqrt{\xi}}{\partial u^3};$$

but from (8) and (26) we obtain

$$\frac{\partial^3 g\left(-\frac{1}{u},\rho\right)}{\partial\left(-\frac{1}{u}\right)^3}=u^4\frac{\partial^3 f(u,\rho)}{\partial u^3}.$$

differentiating in respect to ρ and comparing with the preceding equation, it is finally seen that

$$\frac{\partial^3\sqrt{\xi}}{\partial u^3}=0\qquad\text{or}\qquad\sqrt{\xi}=au^2+bu+c,$$

where a, b and c may depend on ρ. We can now replace (26) by the following

$$\begin{aligned}\xi(u,\rho)&=(au^2+bu+c)^2,\\ \eta(v,\rho)&=v^4\xi\left(-\frac{1}{v},\rho\right)=(cv^2-bv+a)^2.\end{aligned}\tag{27}$$

The process of analytic continuation used in case A does not here show the uniformity of $\partial f/\partial \rho$, since in (24) $\Omega(\rho)$ may have different values for different branches of f. We may however conclude that, when $f(u, \rho)$ and $f_1(u, \rho)$ are two branches of f, there subsists a relation of the form

$$(28) \qquad \frac{\partial f_1(u, \rho)}{\partial \rho} - \frac{\partial f(u, \rho)}{\partial \rho} = h(\rho)\sqrt{\xi(u, \rho)} = h(\rho)(au^2 + bu + c).$$

The condition (26), which may be written

$$(26)' \qquad f'''(u) = \frac{1}{u^4} g'''\left(-\frac{1}{u}\right), \qquad g'''(v) = \frac{1}{v^4} f'''\left(-\frac{1}{v}\right),$$

has a simple geometric interpretation. From (1) we obtain

$$\begin{aligned} dx + idy &= -u^2 f'''(u)du + g'''(v)dv, \\ dx - idy &= f'''(u)du - v^2 g'''(v)dv, \\ dz &= uf'''(u)du + vg'''(v)dv, \end{aligned}$$

and writing the same equations for the parameter values u_1 and v_1,

$$\begin{aligned} dx_1 + idy_1 &= -u_1^2 f'''(u_1)du_1 + g'''(v_1)dv_1, \\ dx_1 - idy_1 &= f'''(u_1)du_1 - v_1^2 g'''(v_1)dv_1, \\ dz_1 &= u_1 f'''(u_1)du_1 + v_1 g'''(v_1)dv_1. \end{aligned}$$

Now make $u_1 = -1/v$, $v_1 = -1/u$, which gives

$$\begin{aligned} dx_1 + idy_1 &= -\frac{1}{v^4} f'''\left(-\frac{1}{v}\right)dv + \frac{1}{u^2} g'''\left(-\frac{1}{u}\right)du \\ &= -g'''(v)dv + u^2 f'''(u)du \end{aligned}$$

by (26)′, so that $dx_1 + idy_1 = -(dx + idy)$. Similarly $(dx_1 - idy_1) = -(dx - idy)$, $dz_1 = -dz$, whence $dx_1 = -dx$, $dy_1 = -dy$, $dz_1 = -dz$, and denoting the constants of integration by $2x_0$, $2y_0$, $2z_0$, we have

$$x_1 - x_0 = -(x - x_0), \quad y_1 - y_0 = -(y - y_0), \quad z_1 - z_0 = -(z - z_0),$$

so that *the surface (1) is symmetrical about the point* x_0, y_0, z_0.

The case B will be examined in detail in § 5.

4. Case A. Detailed discussion.

Case AI: This case is quickly disposed of; for by (12) and (25)′ we have $\partial f/\partial \rho = A_0' + A_1'u + A_2'u^2$, $\partial g/\partial \rho = A_2' - A_1'v + A_0'v^2$, and the comparison with (20) and (21) gives $H = 0$, which was previously recognized as impossible

Case AII: In eq. (13), let $u = u_1$ be a root of $P_4(u, \rho) = 0$; in the vicinity of u_1 we then have an expansion

$$\frac{\partial^3 f}{\partial u^3} = \frac{a_\lambda}{(u-u_1)^\lambda} + \frac{a_{\lambda-1}}{(u-u_1)^{\lambda-1}} + \cdots + \frac{a_1}{u-u_1} + \text{positive powers,}$$
$$a_\lambda \neq 0 \text{ and } 1 \leqq \lambda \leqq 4.$$

If $u_1, a_1, \cdots, a_\lambda$ are independent of ρ, and if this is true for all the roots of $P_4(u, \rho) = 0$, then P_4 is independent of ρ and we find, as in case AI, $\partial f/\partial\rho = A_0 + A_1 u + A_2 u^2$, $\partial g/\partial\rho = A_2 - A_1 v + A_0 v^2$ and $H = 0$.

For the root u_1, we may therefore suppose that either u_1 itself depends on ρ, or in the contrary case, that at least one of $a_1, \cdots, a_\lambda$ is a function of ρ. From the above expansion we obtain

$$\frac{\partial^4 f}{\partial u^3 \partial\rho} = \frac{\lambda a_\lambda \dfrac{du_1}{d\rho}}{(u-u_1)^{\lambda+1}} + \frac{\dfrac{da_\lambda}{d\rho} + (\lambda-1)a_{\lambda-1}\dfrac{du_1}{d\rho}}{(u-u_1)^\lambda} + \cdots + \frac{\dfrac{da_2}{d\rho} + a_1\dfrac{du_1}{d\rho}}{(u-u_1)^2} \tag{29}$$
$$+ \frac{\dfrac{da_1}{d\rho}}{u-u_1} + \text{positive powers of } u - u_1.$$

On the other hand it is necessary, in order that $\partial f/\partial\rho$ be a uniform function of u, that the coefficients of $1/(u-u_1)$, $1/(u-u_1)^2$, $1/(u-u_1)^3$ in (29) all vanish. Consequently $\lambda = 3$ or 4; if $\lambda = 3$, then $du_1/d\rho \neq 0$, and we have another root u_2, which is simple and independent of ρ, as well as the coefficient of $1/(u-u_2)$ in the partial fraction for $\partial^3 f/\partial u^3$. We therefore have, in this case,

$$\xi(u, \rho) = \frac{(u-u_1)^3(u-u_2)}{k(\rho)}, \tag{30}$$

and decomposing its reciprocal into partial fractions

$$\frac{\partial^3 f(u,\rho)}{\partial u^3} = \frac{\dot{k}(\rho)}{(u_1-u_2)^3}\left[\frac{(u_1-u_2)^2}{(u-u_1)^3} - \frac{u_1-u_2}{(u-u_1)^2} + \frac{1}{u-u_1} - \frac{1}{u-u_2}\right].$$

Since the coefficient of $1/(u-u_2)$ must be independent of ρ, we have $k(\rho) = -2c(u_1-u_2)^3$, where c is a constant, and we obtain

$$\frac{\partial f}{\partial\rho} = \frac{c(u_1-u_2)^2\dfrac{du_1}{d\rho}}{u-u_1} + A_0'(\rho) + A_1'(\rho)u + A_2'(\rho)u^2. \tag{31}$$

If however $\lambda = 4$, we may write

$$\xi(u, \rho) = \frac{(u-u_1)^4}{-6k(\rho)} \tag{32}$$

and obtain immediately

$$\frac{\partial f}{\partial \rho} = \frac{k\dfrac{du_1}{d\rho}}{(u - u_1)^2} + \frac{\dfrac{dk}{d\rho}}{u - u_1} + A_0' + A_1'u + A_2'u^2. \tag{33}$$

Similar considerations apply to g, and it follows from (25) that $v_1 = -1/u_1$ and that $\partial f/\partial \rho$ and $\partial g/\partial \rho$ must both be of the form (31) or both of the form (33). In the former case,

$$\eta(v, \rho) = \frac{(v - v_1)^3(v - v_2)}{-2c_1(v_1 - v_2)^3},$$

$$\frac{\partial g}{\partial \rho} = \frac{c_1(v_1 - v_2)^2\dfrac{dv_1}{d\rho}}{v - v_1} + B_0' + B_1'v + B_2'v^2,$$

and computing H, we find

$$H = -\frac{2c(u_1 - u_2)^2\dfrac{du_1}{d\rho}}{(u - u_1)^2} - \frac{2c_1(v_1 - v_2)^2\dfrac{dv_1}{d\rho}}{(v - v_1)^2} - \frac{2v}{1 + uv}\cdot\frac{c(u_1 - u_2)^2\dfrac{du_1}{d\rho}}{u - u_1}$$
$$- \frac{2u}{1 + uv}\cdot\frac{c_1(v_1 - v_2)^2\dfrac{dv_1}{d\rho}}{v - v_1} - A_1' - B_1'$$
$$+ 2\frac{A_1' + B_1' + (A_2' - B_0')u + (B_2' - A_0')v}{1 + uv}.$$

In the expression

$$\frac{\partial}{\partial u}\left(\frac{\xi}{(1 + uv)^2}\frac{\partial H}{\partial u}\right)$$

there is one term containing $1/(v - v_1)$ as a factor, namely

$$\frac{c_1(v_1 - v_2)^2\dfrac{dv_1}{d\rho}}{c(u_1 - u_2)^3}\cdot\frac{\partial}{\partial u}\left(\frac{(u - u_1)^3(u - u_2)}{(1 + uv)^4}\right)\cdot\frac{1}{v - v_1},$$

whereas the right-hand side of (4), substituting H for θ, contains no negative powers of $v - v_1$, owing to the presence of the factor $(v - v_1)^3$ in $\eta(v, \rho)$. Consequently we must have

$$\frac{\partial}{\partial u}\left(\frac{(u - u_1)^3(u - u_2)}{(1 + uv)^4}\right) = 0 \text{ for } v = v_1 = -\frac{1}{u_1},$$

which condition immediately reduces to $u_1^4(u_1 - u_2) = 0$, but neither of these two factors can vanish on account of $du_1/d\rho \neq 0$, so that the case considered is impossible. We now take up the case $\lambda = 4$; then ξ and $\partial f/\partial\rho$ are given by (32) and (33), and we also have

$$\eta(v, \rho) = \frac{(v - v_1)^4}{-6k_1(\rho)},$$

(34)

$$\frac{\partial g}{\partial\rho} = \frac{k_1 \dfrac{dv_1}{d\rho}}{(v - v_1)^2} + \frac{\dfrac{dk_1}{d\rho}}{v - v_1} + B_0' + B_1'v + B_2'v^2.$$

Suppose first that $du_1/d\rho \neq 0$; then also $dv_1/d\rho \neq 0$ since $v_1 = -1/u_1$, and we find

$$H = \frac{-2k\dfrac{du_1}{d\rho}}{(u - u_1)^3} - \frac{\dfrac{dk}{d\rho}}{(u - u_1)^2} - \frac{2k_1\dfrac{dv_1}{d\rho}}{(v - v_1)^2} - \frac{\dfrac{dk_1}{d\rho}}{v - v_1}$$

(35)

$$- \frac{2v}{1 + uv}\left[\frac{k\dfrac{du_1}{d\rho}}{(u - u_1)^2} + \frac{\dfrac{dk}{d\rho}}{u - u_1}\right] - \frac{2u}{1 + uv}\left[\frac{k_1\dfrac{dv_1}{d\rho}}{(v - v_1)^2} + \frac{\dfrac{dk_1}{d\rho}}{v - v_1}\right]$$

$$- A_1' - B_1' + 2\frac{A_1' + B_1' + (A_2' - B_0')u + (B_2' - A_0')v}{1 + uv}.$$

Substituting this expression in (4), we see as before that, $\eta(v, \rho)$ containing the factor $(v - v_1)^4$, no negative powers of $v - v_1$ occur on the right-hand side of the equation, while on the left-hand side we have the terms

(36)

$$\frac{1}{3k}\frac{\partial}{\partial u}\left(\frac{(u - u_1)^4}{(1 + uv)^4}\right)\cdot\left[\frac{k_1\dfrac{dv_1}{d\rho}}{(v - v_1)^2} + \frac{\dfrac{dk_1}{d\rho}}{v - v_1}\right]$$

$$= \frac{4}{3k}\frac{(u - u_1)^3}{(1 + uv)^5}\left[\frac{k_1\dfrac{du_1}{d\rho}}{1 + u_1v} + u_1\frac{dk_1}{d\rho}\right],$$

where, in the last expression, we have replaced v_1 by $-1/u_1$. It is evident that the factor multiplying $1/(1 + u_1v)$ does not vanish for $v = -1/u_1$, so that also this case is impossible.

Next, we assume that $du_1/d\rho = 0$ but $u_1 \neq 0$; then we must have $dk/d\rho \neq 0$. The expressions for $\partial f/\partial\rho$ and $\partial g/\partial\rho$ now become, observing that $v_1 = -1/u_1$ and writing k' and k_1' for $dk/d\rho$ and $dk_1/d\rho$,

$$\frac{\partial f}{\partial\rho} = \frac{k'}{u - u_1} + A_0' + A_1'u + A_2'u^2,$$

(37)

$$\frac{\partial g}{\partial\rho} = \frac{u_1k_1'}{1 + u_1v} + B_0' + B_1'v + B_2'v^2.$$

Substituting these expressions in (25), we obtain

$$\begin{aligned} k' &= u_1^4 k_1', \\ k_1'u_1^3 + B_2' - A_0' &= 0, \\ k_1'u_1^2 - B_1' - A_1' &= 0, \\ k_1'u_1 + B_0' - A_2' &= 0, \end{aligned} \tag{38}$$

and the expression for H becomes

$$\begin{aligned} \frac{1}{u_1k_1'}\cdot H = &-\frac{u_1^3}{(u-u_1)^2} - \frac{u_1}{(1+u_1v)^2} - \frac{2v}{1+uv}\cdot\frac{u_1^3}{u-u_1} \\ &- \frac{2u}{1+uv}\cdot\frac{1}{1+u_1v} - u_1 + 2\frac{u_1+u-u_1^2v}{1+uv}. \end{aligned} \tag{39}$$

Introducing this in (4), we obtain after some algebraic reductions

$$-\frac{1}{k}\cdot\frac{u_1^3}{(1+uv)^2} = -\frac{1}{k_1u_1^4}\cdot\frac{u_1^3}{(1+uv)^2},$$

whence $k = k_1u_1^4$. Writing k instead of k_1, and expressing the B' in terms of the A' by (38), equations (37) give upon integration in respect to ρ

$$\begin{aligned} f(u,\rho) &= \frac{u_1^4k(\rho)}{u-u_1} + A_0(\rho) + A_1(\rho)u + A_2(\rho)u^2, \\ g(v,\rho) &= \frac{u_1k(\rho)}{1+u_1v} + A_2(\rho) - u_1k(\rho) + [u_1^2k(\rho) - A_1(\rho)]v \\ &\qquad + [A_0(\rho) - u_1^3k(\rho)]v^2. \end{aligned} \tag{40}$$

The family of minimal surfaces defined by these values of f and g is a special case of one determined later (case *B2*).

Finally, we have to discuss the case when $u_1 = 0$; then

$$\frac{\partial^3 f}{\partial u^3} = \frac{-6k(\rho)}{u^4}; \qquad f = \frac{k}{u} + A_0 + A_1u + A_2u^2,$$

$$\frac{\partial f}{\partial \rho} = \frac{k'}{u} + A_0' + A_1'u + A_2'u^2;$$

$$\frac{\partial g}{\partial \rho} = v^2\frac{\partial f\left(-\frac{1}{v},\rho\right)}{\partial \rho} = -k'v^3 + A_2' - A_1'v + A_0'v^2,$$

$$g(v,\rho) = -kv^3 + A_2 - A_1v + A_0v^2 + g_0(v),$$

$$\frac{\partial^3 g}{\partial v^3} = -6k(\rho) + \frac{d^3g_0(v)}{dv^3},$$

$$\frac{1}{k'}H = -\frac{1}{u^2} - 3v - \frac{2v}{1+uv}\cdot\frac{1}{u} + \frac{2uv^3}{1+uv}$$
$$= -\frac{1}{u^2} - v^2 - 2\frac{v}{u},$$

and substituting this value of H in (4), we find that this equation reduces to

$$\frac{d^3g_0(v)}{dv^3} = 0,$$

so that we obtain the following expressions for f and g,

$$(41)\qquad \begin{aligned} f(u,\rho) &= \frac{k(\rho)}{u} + A_0(\rho) + A_1(\rho)u + A_2(\rho)u^2, \\ g(v,\rho) &= -k(\rho)v^3 + A_2(\rho) - A_1(\rho)v + A_0(\rho)v^2. \end{aligned}$$

We have here assumed $g_0(v) = 0$, which is legitimate, since the addition to $g(v, \rho)$ of a polynomial of the second degree with coefficients independent of ρ merely signifies a simultaneous translation, of constant magnitude and direction, of the entire family (1). The family defined by (41) is again a special case of one considered later (case *B3*).

Case AIII: We begin by examining the cases where, in (16) and (18),

$$(42)\qquad ad - bc = 0.$$

First, suppose $a = b = c = d = 0$; then (20) and (25)′ give $\partial f/\partial\rho = B_1u^2 + C_1u + D_1$, $\partial g/\partial\rho = D_1v^2 - C_1v + B_1$, and by (21) $H = 0$, which was seen to be impossible. Second, suppose $b = d = 0$, but either $a \neq 0$ or $c \neq 0$; then the second equation (18) gives

$$\frac{\partial g}{\partial\rho} = A_4v^2 + B_4v + C_4 + \frac{D_4}{v},$$

whence, by (25),

$$\frac{\partial f}{\partial\rho} = A_4 - B_4u + C_4u^2 - D_4u^3,$$

and according as $a \neq 0$ or $a = 0$, the first or second equation (16) gives $\xi = P_4(u, \rho)$ or $\xi = P_5(u, \rho)$. But at a zero of ξ, $\partial f/\partial\rho$ becomes infinite according to (8) and thus contradicts the first equation (16), unless this zero and all the corresponding coefficients in the partial fraction for $1/\xi$ are independent of ξ, that is, unless $\partial\xi/\partial\rho = 0$. Then, however, (8) and (25)′ give $\partial f/\partial\rho = A_0' + A_1'u + A_2'u^2$, $\partial g/\partial\rho = A_2' - A_1'v + A_0'v^2$, and we have again $H = 0$. The case $c = d = 0$ is treated in the same way. The case $d = 0$, $b \neq 0$ and $c \neq 0$ is impossible, since then $ad - bc \neq 0$.

There now remains the case $d \neq 0$, in which we obtain from (16), (18) and (42)

$$(43) \qquad (bu - d)\frac{\partial f}{\partial \rho} = (bA_2 - dA_1)u^3 + \cdots + (bD_2 - dD_1),$$

$$(44) \qquad (cv - d)\frac{\partial g}{\partial \rho} = (cA_4 - dA_3)v^3 + \cdots + (cD_4 - dD_3),$$

and applying (25) to (44), we find

$$(45) \qquad (du + c)\frac{\partial f}{\partial \rho} = (cA_4 - dA_3) \cdot \frac{1}{u} - (cB_4 - dB_3) + (cC_4 - dC_3)u - (cD_4 - dD_3)u^2.$$

If $b = 0$, then (42) gives $a = 0$, (43) and (45) show that $A_1 = 0$, so that the first of (16) gives $\partial f/\partial \rho$ as a polynomial of the second degree in u, whence we conclude as before that $H = 0$. If $b \neq 0$, and $bu - d$ is a factor of the right-hand side in (43), we arrive at the same conclusion. Finally, when the right-hand side in (43) is not divisible by $bu - d$, the comparison of (43) and (45) gives

$$b/d = -d/c \qquad \text{or} \qquad bc + d^2 = 0,$$

$$cA_4 - dA_3 = 0, \qquad bA_2 - dA_1 = 0,$$

whence, by (43),

$$(46) \qquad \frac{\partial f}{\partial \rho} = \frac{P_2(u, \rho)}{bu - d}.$$

The combination of $bc + d^2 = 0$ with (42) gives $a + d = 0$, and by (46), the first of equations (16) may now be written

$$(bu - d)\frac{\partial \xi}{\partial u} - 4b\xi = \frac{P_2(u, \rho)}{bu - d} - A_1u^3 - B_1u^2 - C_1u - D_1.$$

Integrating this linear differential equation of the first order, we obtain

$$\xi = (bu - d)^4\left[\int\left(\frac{P_2(u, \rho)}{(bu - d)^5} - \frac{A_1u^3 + B_1u^2 + C_1u + D_1}{(bu - d)^4}\right)du + C(\rho)\right],$$

or performing the integration

$$\xi = P_4(u, \rho) - \frac{A_1}{b^3}(bu - d)^4 \log(bu - d).$$

Substituting this expression back into the first equation (16), we see that, since $a = -d$, $\partial f/\partial \rho = P_3(u, \rho)$, so that $\partial f/\partial \rho$ has no pole at $-d/b$ contrary to our hypothesis.

We may now assume that $ad - bc \neq 0$. Then equations (16) may be solved for ξ and $\partial\xi/\partial u$, giving

$$
(47)\quad
\begin{aligned}
\xi(u,\rho) &= -\frac{1}{4(ad-bc)}\left\{[bu^2+(a-d)u-c]\frac{\partial f}{\partial\rho}-(a+bu)\right.\\
&\qquad\left.\times(A_2u^3+\cdots+D_2)+(c+du)(A_1u^3+\cdots+D_1)\right\},\\
\frac{\partial\xi(u,\rho)}{\partial u} &= \frac{1}{ad-bc}\left\{(d-bu)\frac{\partial f}{\partial\rho}-d(A_1u^3+\cdots+D_1)\right.\\
&\qquad\left.+\,b(A_2u^3+\cdots+D_2)\right\},
\end{aligned}
$$

whence, differentiating the first equation in respect to u and substituting in the second, we obtain the following linear differential equation for $\partial f/\partial\rho$:

$$
(48)\quad [bu^2+(a-d)u-c]\frac{\partial^2 f}{\partial u\partial\rho}+[-2bu+(a+3d)]\frac{\partial f}{\partial\rho}+P_2(u,\rho)=0,
$$

where the coefficients in the polynomial of the second degree in u, $P_2(u,\rho)$, are of no particular interest to us. From (18), we get expressions for η and $\partial\eta/\partial v$ similar to (47), and the differential equation for $\partial g/\partial\rho$:

$$
(49)\quad [cv^2+(a-d)v-b]\frac{\partial^2 g}{\partial v\partial\rho}+[-2cv+(a+3d)]\frac{\partial g}{\partial v}+Q_2(v,\rho)=0.
$$

From (47), and the similar expression for $\eta(v,\rho)$ it follows that, since $\partial f(u,\rho)/\partial\rho$ and $\partial g(v,\rho)/\partial\rho$ are uniform functions of u and v respectively, the same is also the case with $\xi(u,\rho)$ and $\eta(v,\rho)$.

Let us first suppose that the roots α and β of $bu^2+(a-d)u-c=0$ are distinct; the roots $-1/\alpha$ and $-1/\beta$ of $cv^2+(a-d)v-b=0$ are then also distinct. According to (17), the only singularities of ξ are α and β (and infinity, if both α and β are finite). Suppose that α is finite, then, since α and β are distinct, α is a regular singular point for the linear differential equation (17), and since ξ is uniform, α can only be a pole, in the vicinity of which we have the expansion

$$
\xi=\frac{a_\lambda}{(u-\alpha)^\lambda}+\frac{a_{\lambda-1}}{(u-\alpha)^{\lambda-1}}+\cdots+\frac{a_1}{u-\alpha}+\cdots,\quad \lambda>0,\ a_\lambda\neq 0.
$$

But at a pole of ξ, its reciprocal $\partial^3 f/\partial u^3$ is holomorphic, and consequently also $\partial f/\partial\rho$ Substituting the expansion of ξ in the first equation (16), we obtain

$$
a+b\alpha=0,\qquad -\lambda b-4b=0,
$$

whence, since $\lambda>0$, $b=a=0$ and $ad-bc=0$ contrary to our assumption. Thus ξ has no poles at finite distance, but is an entire function, and by the first of (16) the same is true of $\partial f/\partial\rho$. A similar reasoning shows that $\partial g/\partial\rho$ is an entire function of v, and (25) and (25)′ prove that either function has a pole of no higher than the second order at infinity, that is, $\partial f/\partial\rho$ and $\partial g/\partial\rho$ are polynomials of the second degree. Now (25) and (21) finally give $H = 0$, which is impossible. As is readily seen, the preceding argument holds even when $bu^2 + (a - d)u - c$ vanishes identically.

There remains the case when $bu^2 + (a - d)u - c = 0$ has a finite double root α, so that

$$(a - d)^2 + 4bc = 0. \tag{50}$$

If ξ has a pole at α, the preceding argument holds; we must therefore suppose that α is an essential singularity of ξ. Then α is also an essential singularity of $\partial f/\partial\rho$, for in the contrary case, the first equation (16), regarded as a linear differential equation in ξ, would show that α were at most a pole of ξ.

Now transform (49) by making $v = -1/u$; as (25) gives

$$u^2 \frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial\rho} = \frac{\partial f(u, \rho)}{\partial\rho},$$

$$2u \frac{\partial g\left(-\frac{1}{u}, \rho\right)}{\partial\rho} + \frac{\partial^2 g\left(-\frac{1}{u}, \rho\right)}{\partial\left(-\frac{1}{u}\right)\partial\rho} = \frac{\partial^2 f(u, \rho)}{\partial\rho},$$

we obtain

$$[bu^2 + (a - d)u - c]\frac{\partial^2 f}{\partial u \partial\rho} + [-2bu - (3a + d)]\frac{\partial f}{\partial\rho} - u^2 Q\left(-\frac{1}{u}, \rho\right) = 0.$$

This equation must be identical with (48), since otherwise the combination of both would show that $\partial f/\partial\rho$ were holomorphic at α; consequently

$$a + d = 0,$$

and this equation, combined with (50), gives $ad - bc = 0$ contrary to our hypothesis.

5. Case B. Detailed discussion.

This case will be subdivided according to the various forms which the expressions (27) for ξ and η can take, in the following manner

Case B1: $a \neq 0$, $b^2 - 4ac \neq 0$,

Case B2 $a \neq 0$, $b^2 - 4ac = 0$, $c \neq 0$,

Case B3: $a \neq 0$, $b^2 - ac = 0$, $c = 0$,

Case B4: $a = c = 0$.

The only remaining possibility, $a = 0$ and $c \neq 0$, reduces, by interchanging u and f with v and g, to B1 or B3 according as $b \neq 0$ or $b = 0$.

Case B1.—We have, denoting by u_1 and u_2 the roots of $au^2 + bu + c = 0$,

$$\xi(u, \rho) = \frac{(u - u_1)^2 (u - u_2)^2}{k}, \qquad u_1 \neq u_2,$$

and by (8) and (27)

$$\begin{aligned} \frac{\partial^3 f}{\partial u^3} &= \frac{k}{(u - u_1)^2 (u - u_2)^2}, \\ \frac{\partial^3 g}{\partial v^3} &= \frac{k}{(1 + u_1 v)^2 (1 + u_2 v)^2}. \end{aligned} \tag{51}$$

Decomposing into partial fractions, we find

$$\begin{aligned} \frac{\partial^3 f}{\partial u^3} &= \frac{k}{(u_1 - u_2)^2} \left[\frac{1}{(u - u_1)^2} + \frac{1}{(u - u_2)^2} \right] \\ &\qquad - \frac{2k}{(u_1 - u_2)^3} \left[\frac{1}{u - u_1} - \frac{1}{u - u^2} \right], \\ \frac{\partial^3 g}{\partial v^3} &= \frac{k}{(u_1 - u_2)^2} \left[\frac{u_1^2}{(1 + u_1 v)^2} + \frac{u_2^2}{(1 + u_2 v)^2} \right] \\ &\qquad - \frac{2k u_1 u_2}{(u_1 - u_2)^3} \left[\frac{u_1}{1 + u_1 v} - \frac{u_2}{1 + u_2 v} \right], \end{aligned} \tag{52}$$

whence

$$\begin{aligned} \frac{\partial^4 f}{\partial u^3 \partial \rho} &= \frac{2k}{(u_1 - u_2)^2} \left[\frac{\dfrac{du_1}{d\rho}}{(u - u_1)^3} + \frac{\dfrac{du_2}{d\rho}}{(u - u_2)^3} \right] \\ &\quad + \left[\frac{\dfrac{dk}{d\rho}}{(u_1 - u_2)^2} - \frac{2k\left(2\dfrac{du_1}{d\rho} - \dfrac{du_2}{d\rho}\right)}{(u_1 - u_2)^3} \right] \cdot \frac{1}{(u - u_1)^2} \\ &\quad + \left[\frac{\dfrac{dk}{d\rho}}{(u_1 - u_2)^2} - \frac{2k\left(\dfrac{du_1}{d\rho} - 2\dfrac{du_2}{d\rho}\right)}{(u_1 - u_2)^3} \right] \cdot \frac{1}{(u - u_2)^2} \\ &\quad - 2\frac{d}{d\rho}\left(\frac{k}{(u_1 - u_2)^3} \right) \cdot \left[\frac{1}{u - u_1} - \frac{1}{u - u_2} \right] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f}{\partial \rho} &= \frac{k}{(u_1 - u_2)^2} \left[\frac{du_1}{d\rho} \log (u - u_1) + \frac{du_2}{d\rho} \log (u - u_2) \right] \\ &\quad - \left[\frac{\dfrac{dk}{d\rho}}{(u_1 - u_2)^2} - \frac{2k\left(2\dfrac{du_1}{d\rho} - \dfrac{du_2}{d\rho}\right)}{(u_1 - u_2)^3} \right] (u - u_1) \log (u - u_1) \end{aligned} \tag{53}$$

$$(53)\qquad -\left[\frac{\dfrac{dk}{d\rho}}{(u_1-u_2)^2}-\frac{2k\left(\dfrac{du_1}{d\rho}-2\dfrac{du_2}{d\rho}\right)}{(u_1-u_2)^3}\right](u-u_2)\log(u-u_2)$$
$$-\frac{d}{d\rho}\left(\frac{k}{(u_1-u_2)^3}\right)[(u-u_1)^2\log(u-u_1)-(u-u_2)^2\log(u-u_2)]$$
$$+A_0'+A_1'u+A_2'u^2.$$

Making an analytic continuation around $u = u_1$ and applying (28), we obtain

$$\frac{k}{(u_1-u_2)^2}\frac{du_1}{d\rho}-[\cdots](u-u_1)-\frac{d}{d\rho}\left(\frac{k}{(u_1-u_2)^3}\right)\cdot(u-u_1)^2$$
$$=h(u-u_1)(u-u_2),$$

and setting $u = u_1$, this reduces to $du_1/d\rho = 0$. Similarly $du_2/d\rho = 0$, so that u_1 and u_2 are independent of ρ. We may now show that this case reduces to case *B4* by changing our coördinate axes.

For this purpose we shall use a theorem which is obtained immediately by combining the formulas in Darboux's Théorie des Surfaces, vol. I, p. 391, with those p. 46 and following, namely:

The substitution of U, V, $F(U)$ and $G(V)$ for u, v, $f(u)$ and $g(v)$ in equations (1), where

$$(54)\qquad \begin{gathered} U=\frac{lu-m}{nu+p},\qquad V=\frac{pv-n}{mv+l},\\ \frac{d^3F(U)}{dU^3}=\frac{(nu+p)^4}{(lp+mn)^2}\frac{d^3f(u)}{du^3},\qquad \frac{d^3G(V)}{dV^3}=\frac{(mv+l)^4}{(lp+mn)^2}\frac{d^3g(v)}{dv^3},\end{gathered}$$

and l, m, n, p are constants ($lp + mn \neq 0$), is equivalent to the orthogonal coördinate transformation defined by

$$(55)\qquad \begin{aligned} X+iY&=\frac{l^2(x+iy)-m^2(x-iy)+2lmz}{lp+mn},\\ X-iY&=\frac{-n^2(x+iy)+p^2(x-iy)+2npz}{lp+mn},\\ Z&=\frac{-ln(x+iy)-mp(x-iy)+(lp-mn)z}{lp+mn}.\end{aligned}$$

In the case under consideration, make $l = 1$, $m = u_1$, $n = 1$, $p = -u_2$; since u_1 and u_2 are independent of ρ, the same is the case with the coefficients in (55), so that every individual surface of our family is referred to the *same* new system of coördinates X, Y, Z. From (54) we obtain

$$U = \frac{u - u_1}{u - u_2}, \qquad V = -\frac{1 + u_2 v}{1 + u_1 v},$$

$$\frac{\partial^3 F}{\partial U^3} = \frac{(u - u_2)^4}{(u_1 - u_2)^2} \cdot \frac{k}{(u - u_1)^2 (u - u_2)^2} = \frac{k}{(u_1 - u_2)^2} \cdot \frac{1}{U^2},$$

$$\frac{\partial^3 G}{\partial V^3} = \frac{(1 + u_1 v)^4}{(u_1 - u_2)^2} \cdot \frac{k}{(1 + u_1 v)^2 (1 + u_2 v)^2} = \frac{k}{(u_1 - u_2)^2} \cdot \frac{1}{V^2},$$

or writing

$$K(\rho) = -\frac{2k(\rho)}{(u_1 - u_2)^2},$$

$$\frac{\partial^3 F}{\partial U^3} = -\frac{K(\rho)}{2U^2}, \qquad \frac{\partial^3 G}{\partial V^3} = -\frac{K(\rho)}{2V^2},$$

which are precisely the equations of definition of case *B4*.

Case B2.—We have

$$\frac{\partial^3 f(u, \rho)}{\partial u^3} = -\frac{6ku_1^4}{(u - u_1)^4}, \qquad \frac{\partial^3 g(v, \rho)}{\partial v^3} = -\frac{6ku_1^4}{(1 + u_1 v)^4}, \qquad u_1 \neq 0,$$

whence

$$f(u, \rho) = \frac{ku_1^4}{u - u_1} + A_0 + A_1 u + A_2 u^2,$$

$$g(v, \rho) = \frac{ku_1}{1 + u_1 v} + B_0 + B_1 v + B_2 v^2.$$

If first $du_1/d\rho \neq 0$, the reasoning which led to formulas (35) and (36) is still valid, and yields the same contradiction as in case *A*. Therefore u_1 is independent of ρ, and the present case may be reduced to case *B3* by making $l = 1$, $m = u_1$, $n = 0$, $p = 1$ in (54) and (55), whence

$$U = u - u_1, \qquad V = \frac{v}{1 + u_1 v},$$

$$\frac{\partial^3 F}{\partial U^3} = -\frac{6ku_1^4}{(u - u_1)^4} = -\frac{6ku_1^4}{U^4},$$

$$\frac{\partial^3 G}{\partial V^3} = -(1 + u_1 v)^4 \cdot \frac{6ku_1^4}{(1 + u_1 v)^4} = -6ku_1^4,$$

or by writing $K(\rho) = k(\rho) \cdot u_1^4$,

$$\frac{\partial^3 F}{\partial U^3} = -\frac{6K(\rho)}{U^4}, \qquad \frac{\partial^3 G}{\partial V^3} = -6K(\rho),$$

that is, the equations of definition of case *B3*.

Case B3.—We now have

$$\xi(u, \rho) = \frac{u^4}{-6k(\rho)}, \qquad \eta(v, \rho) = -\frac{1}{6k(\rho)},$$

$$f(u, \rho) = \frac{k}{u} + A_0 + A_1 u + A_2 u^2,$$

$$g(v, \rho) = -kv^3 + B_0 + B_1 v + B_2 v^2,$$

and (24) gives, writing $\Omega(\rho)\sqrt{\xi(u, \rho)} = l'(\rho) \cdot u^2$,

$$B_0' - A_2' = l', \qquad B_1' + A_1' = 0, \qquad B_2' - A_0' = 0,$$

$$H = -k'\left(\frac{1}{u^2} + v^2 + 2\frac{v}{u}\right) - \frac{2l'u}{1 + uv}.$$

As is readily seen, this H satisfies (4); the corresponding family of minimal surfaces is

$$x + iy = -6k(\rho)\left(\frac{1}{u} + v\right),$$

(56) $$x - iy = 2k(\rho)\left(\frac{1}{u^3} + v^3\right) - l(\rho),$$

$$z = 3k(\rho)\left(\frac{1}{u^2} - v^2\right).$$

None of these surfaces are real; in the special case $l = 0$, we obtain the surfaces defined by (41). The elimination of the parameters u and v in (56) gives

(57) $$\frac{1}{432k^2}(x + iy)^4 + (x + iy)(x - iy + l) + z^2 = 0.$$

The only singularity of this surface is the double point $x = -\frac{1}{2}l$, $y = -\frac{1}{2}li$, $z = 0$, which is also a point of symmetry of the surface. The only straight line on the surface is $x + iy = 0$, $z = 0$.

Case B4.—We may write

$$\xi(u, \rho) = -\frac{2u^2}{k(\rho)}, \qquad \eta(v, \rho) = -\frac{2v^2}{k(\rho)},$$

$$f(u, \rho) = \frac{k}{2}(u \log u - u) + A_0 + A_1 u + A_2 u^2,$$

$$g(v, \rho) = \frac{k}{2}(v \log v - v) + B_0 + B_1 v + B_2 v^2,$$

and (24) gives, $l(\rho)$ being arbitrary,

$$B_0' - A_2' = 0, \qquad B_1' + A_1' = -k' - l', \qquad B_2' - A_2' = 0.$$

The equation (4) is satisfied, and the corresponding minimal surfaces are catenoids with the z-axis as common axis of revolution; for (1) gives

$$
\begin{aligned}
x + iy &= \frac{k}{2}\left(u + \frac{1}{v}\right),\\
x - iy &= \frac{k}{2}\left(\frac{1}{u} + v\right), \qquad (58)\\
z &= -\frac{k}{2}(\log u + \log v) + l,
\end{aligned}
$$

whence

$$x^2 + y^2 = \frac{k^2}{4}\left(uv + 2 + \frac{1}{uv}\right), \qquad \sqrt{x^2 + y^2} = \frac{k}{2}\left(\sqrt{uv} + \frac{1}{\sqrt{uv}}\right),$$

$$\sqrt{uv} = e^{-(z-l)/k},$$

$$\sqrt{x^2 + y^2} = \frac{k}{2}\left(e^{(z-l)/k} + e^{-(z-l)/k}\right), \qquad (59)$$

$k = k(\rho)$ and $l = l(\rho)$ being arbitrary functions of ρ.

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